Formalizing a Reasoning Strategy in Symbolic Approach to Differential Equations[1]

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Abstract

The following reasoning schema is used in symbolic approach to differential equations. Given a family of parameterized objects, for example, $\dot{x} = f(x, u)$ where $x = (x_1, \cdots, x_n)$ and parameters $u = (u_1, \cdots, u_l)$, and a first order property $P(u_1, \cdots, u_l)$ which may be a conjecture, how to automatically derive the conditions on the parameters $u_1, \cdots, u_l$ that characterize the objects having (or not having) the property P in the family? That is, how to find a theory Γ such that $\Gamma \vdash P$ or $\Gamma \vdash \neg P$?

Let us see the *center-focus* problem, which is the key to the second part of Hilbert's 16th problem.

Center-focus problem: Consider differential systems of *center-focus* type

$$\dot{x} = \lambda x + y + P(x, y), \qquad \dot{y} = -x + \lambda y + Q(x, y),$$

where $P(x, y)$ and $Q(x, y)$ are homogeneous polynomials of degree $k \geq 2$ and with indeterminate coefficients $u = (u_1, \cdots, u_l)$. As explained by Wang (1991), Lu and Ma (2000), one can compute the *Liapunov function* $F(x, y)$ and polynomials $\eta_2, \eta_4, \cdots, \eta_{2i+2}, \cdots$ in $u_1, \cdots, u_l$ such that

$$\frac{dF(x, y)}{dt} = \eta_2(x^2 + y^2) + \eta_4(x^2 + y^2)^2 + \cdots + \eta_{2i+2}(x^2 + y^2)^{2i} + \cdots,$$

where η_{2i+2} are called *focal values*.

The origin $(0, 0)$ is a *fine focus of order* k if

$$\eta_{2i} = 0 \text{ for } 1 \leq i \leq k, \text{ but } \eta_{2k+2} \neq 0.$$

A fine focus of infinite order is called a *center*. Deciding the origin to be a center or a fine focus of finite order is the key to construct the limit cycles for the above differential systems.

A successful strategy is to compute first few focal values $\eta_2, \cdots, \eta_{2i}$ for $i < j < k$, and guess some simple conditions on the parameters $u_1, \cdots, u_l$, $\Gamma = \{f_1 = 0, \cdots, f_m = 0\}$, to make these focal values be zero, (i.e., $\eta_2 = 0, \cdots, \eta_{2i} = 0 \in Th(\Gamma)$) and then simplify the differential system using these conditions. Then, compute $\eta_{2i+2}, \cdots, \eta_{2j}$ from the simplified differential system, and determine that

1. if $\eta_{2i+2} = 0 \in Th(f_1 = 0, \cdots, f_m = 0)$, then check $\eta_{2i+4} = 0, \cdots$;
2. if $\eta_{2i+2} = 0 \notin Th(f_1 = 0, \cdots, f_m = 0)$ and $\eta_{2i+2} \neq 0 \notin Th(f_1 = 0, \cdots, f_m = 0)$ then make a guess $h = 0$, consistent with $\{f_1 = 0, \cdots, f_m = 0\}$, such that $\eta_{2i+2} = 0$ and add $h = 0$ into $\{f_1 = 0, \cdots, f_m = 0\}$;

   Remark: Here, a guess $h = 0$ can be got from a factor h_j of $\eta_{2i+2} = h_1 \cdots h_k$, as $\eta_{2i+2} = 0$ iff $h_1 = 0$ or $\cdots$ or $h_k = 0$.
3. if $\eta_{2i+2} \neq 0 \in Th(f_1 = 0, \cdots, f_m = 0)$, then we have to revise our guess Γ to have a new guess Γ' and simplify the differential system again.

Keep going on, until that the goal is proved or disproved.

In this talk, a computation model involving the computation of limits of formal theory sequences is proposed. It is called *procedure scheme*. It provides an approach to build a new theory by the limit of some sequence of formal theories. A convergent procedure scheme called DISCOVER is defined in the algebraically closed fields (ALC). We can formalize the above strategy using the procedure scheme DISCOVER.

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