On "Good" Basis of Polynomial Ideals

Wu Wen-tsun

[1] (*Institute of Systems Science, Academia Sinica, Beijing* 100080, *China*)

Let PS be a finite polynomial set in the polynomial ring $R = K[x_1, \cdots, x_n]$, K being a coefficient field of characteristic 0. Then there are two important problems connected with PS, viz.:

Problem P1. Determine the totality of solutions of $PS = 0$ in all conceivable extension fields of K, to be denoted by $Zero(PS)$ in what follows.

Problem P2. For the ideal $Id(PS)$ with basis PS determine some kind of "good" basis which will enjoy some *nice* properties to be precised.

For the Problem P1 the present author has given a method of determining completely $Zero(PS)$ in the following manner:

Arrange the variables in the natural order then a non-constant polynomial $P \in R$ may be written in the canonical form below:

$$P = I_0 * x_c^d + I_1 * x_c^{d-1} + \cdots + I_d,$$

in which I_j are all either constants or polynomials in $x_1, \cdots, x_{c-1}$ alone with *initial* $I_0 \neq 0$. With respect to *class* c and *degree* d we may introduce a partial ordering $\prec$ for all non-zero polynomials in R with non-zero constant polynomials in the lowest ordering. Consider now some polynomial set either consisting of a single non-zero constant polynomoal or the polynomials may be so arranged with classes all positive and steadily increasing. Call such polynomial sets *ascending sets*. Then we may introduce a partial ordering $\prec$ among all such ascending sets with the trivial ones consistng of a single non-zero constant polynomial in the lowest ordering. For a finite polynomial set any ascending set wholly contained in it and of lowest ordering is called a *basic set* of the given polynomial set. A partial ordering among all finite polynomial sets may then be unambiguously introduced according to their basic sets.

For any finite polynomial set $PS \subset R$ consider now the scheme (S) below:

$$\begin{array}{rllllllll} PS = & PS^0 & PS^1 & \cdots & PS^i & \cdots & PS^m & & \\ & BS^0 & BS^1 & \cdots & BS^i & \cdots & BS^m & = & CS \\ & RS^0 & RS^1 & \cdots & RS^i & \cdots & RS^m & = & \emptyset. \end{array} \tag{S}$$

In the scheme (S) each BS^i is a basic set of PS^i, each RS^i is the set of non-zero remainders, if any, of polynomials in $PS^i \setminus BS^i$ with respect to BS^i, and $PS^{i+1} = PS \cup BS^i \cup RS^i$ if RS^i is non-empty. It is easily proved that the sequences in the scheme should terminate at certain stage m with $RS^m = \emptyset$. The corresponding basic set $BS^m = CS$ is then called a *characteristic set* of the given polynomial system PS. The *zero-set* of $PS, Zero(PS)$, which is the collection of common zeros of all polynomials in PS, is closely connected with that of CS by the *Well-Ordering Principle* in the form below:

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$$Zero(PS) = Zero(CS/IP) \cup Zero(PS \ \cup \ \{IP\}),$$

in which IP is the product of all initials of polynomials in CS and $Zero(CS/IP) = Zero(CS) \setminus Zero(IP)$.

Now $PS \cup \{IP\}$ is easily seen to be a polynomial set of lower ordering than PS. If we apply the Well-Ordering Principle to $PS \cup \{IP\}$ and proceed further and further in the same way we should stopped in a finite number of steps and arrived at the following

Zero-Decomposition Theorem. For any finite polynomial system PS there is an algorithm which will give in a finite number of steps a finite set of asc-sets CS^s with initial-product IP^s such that

$$Zero(PS) = \bigcup_s Zero(CS^s \ / \ IP^s). \tag{Z}$$

Now CS^s are all ascending sets. Hence all zero-sets $Zero(CS^s)$ and all $Zero(CS^s / \ IP^s)$ may be considered as well-determined in some natural sense. The formula (Z) gives thus actually an explicit determination of $Zero(PS)$ for all finite polynomial systems PS which serves for the solving of arbitrary systems of polynomial equations.

The above method of solving arbitrary systems of polynomial equations has been extended to arbitrary systems of *algebrico-differential* equations, either ordinary or partial ones, which will be explained below.

Let $y, u_j, j \in J$, be infinitely differentiable functions in independent variables $X = \{x_k, k = 1, \cdots .n\}$. A polynomial in various derivatives of y and u_j with respect to x_k with coefficients in the differential field of rational functions of X will be called an *algrbrico-differential polynomial*. Suppose given a finite system of such polynomials $DPS = \{DP_i \mid i \in I\}$. Let us consider the associated system of partial differential equations of y with u_j supposed known:

$$DPS = 0, \quad or \quad DP_i = 0, \ i \in I.$$

Our problem is to determine the integrability conditions for y to be solvable in terms of x_k, u_j and in affirmative case to determine the set of all possible formal solutions of y.

Criteria and even algorithmic methods of solving the above problem were known in quite remote times for which we may cite particularly Riquier, Janet, and E.Cartan. In recent years J.F.Pommaret had given a systematic *formal intrinsic* way of treatment and had published several voluminous treatises. On the other hand, the present author had given an alternative method in following essentially the steps of Riquier and Janet, cf. [W1]. The method consists in first extending naturally the notions of ascending sets, basic sets, remainders, et al in the ordinary case to the present algebrico-differential case. Orderings among all derivatives and then partial orderings may then successively intoduced among all algebrico-differential polynomials, all differential-ascending sets, and finally all systems of algebrico-differential polynomial sets, somewhat analogous to the ordinary case.

For any system DPS of algebrico-differential polynomials we may form a scheme (dS) analogous to the scheme in the ordinary case as shown below:

$$\begin{array}{lccccccccccccc}
DPS = & DPS^0 & & DPS^1 & & \cdots & & DPS^i & & \cdots & & DPS^m & & \\
& DBS^0 & & DBS^1 & & \cdots & & DBS^i & & \cdots & & DBS^m & = & DCS \\
& DRIS^0 & & DRIS^1 & & \cdots & & DRIS^i & & \cdots & & DRIS^m & = & \emptyset \\
& DCPS^0 & \cup & DCPS^1 & \cup & \cdots & \cup & DCPS^i & \cup & \cdots & \cup & DCPS^m & = & DCPS
\end{array} \tag{dS}$$

In the scheme (dS) DPS is the given algebrico-differential polynomial set. For each i, DBS^i is a differential basic set of DPS^i. The set $DRIS^i$ is the union of two parts. One is the set of all possible non-zero remainders formed from differential polynomials in $DPS^i \setminus DBS^i$ with respect to DBS^i, while the other is the set of integrability differential polynomials formed from certain pairs of differential polynomials in DPS^i, so far they contain actually y or its derivatives. Such pairs will be determined by notions of *multiplicativity* and *non-multiplicativity* due to Riquier and Janet. On the other hand those containing no y or its derivatives but containing possibly u_j or their derivatives will form a set of *compatibility differential polynomials* for which the vanishing will form the *compatibility conditions* in order that the given set of equations $DPS = 0$ will have solutions. In case $DRIS^i$ is non-empty, then the union $DPS \cup DBS^i \cup DRIS^i$ will form the next differential polynomial set DPS^{i+1}.

As in the ordinary case the sequences will terminate at a certain stage m with $DRIS^m = \emptyset$. The corresonding differential basic set $DBS^m = DCS$ is then called a *differential characteristic set* of the given differential polynomial set DPS. The union $DCPS$ of all sets $DCPS^i, i = 1, \cdots, m$, will form the totality of all possible compatibility differential polynomials whose vanishing form the compatibility conditions to guarantee the existence of solutions of the partial differential equations $DPS = 0$.

As in the ordinary case the above will lead finally to the formation of the totallity of formal solutions of the given system of algebrico-differential equations under suitable initial datas for which we refer to the paper [W1]

Let us now consider the particular case for which the differential polynomials in DPS are all *linear* with *constant coefficients.* For each tuple of non-negative integers $\mu = (i_1, \cdots, i_n)$ let us write $\|\mu\|$ for $i_1 + \cdots i_n$ and make the correspondence

$$\textit{partial derivative } \frac{\partial^{\|\mu\|}}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}} \longleftrightarrow \textit{Monomial } x_1^{i_1} * \cdots * x_n^{i_n}$$

Then the partial differentiation of a derivative with respect to some x_j will correspond to the multiplication of the correspoding monomial with the variable x_j. In this way a differential polynomial set DPS consisting of only constant-coefficients linear differential polynomials will become under the above correspondence a polynomial set PS in the ordinary sense. The scheme (dS) will then be turned into some scheme (G) for PS somewhat of the following form:

$$\begin{array}{lllllllll} PS = & PS^0 & PS^1 & \cdots & PS^i & \cdots & PS^m & & \\ & WS^0 & WS^1 & \cdots & WS^i & \cdots & WS^m & = & GS \\ & IS^0 & IS^1 & \cdots & IS^i & \cdots & IS^m & = & \emptyset. \end{array} \tag{G}$$

In the above scheme (G) the WS^i are certain subsets of PS^i enjoying some well-arranged properties and each IS^i are consisting of remainders of polynomials in $PS^i \setminus WS^i$ with respect to WS^i as well as those determined from certain pairs of polynomials in WS^i determined by the notions of multicativity and non-multiplicativity of Riquier and Janet. The union of WS^i, IS^i and eventually PS^0 will then be PS^{i+1} so far $IS^i \neq \emptyset$. It turns out that the final set GS is a basis of the given ideal $Id(PS)$ and possesses many nice properties. It turns out that this basis GS is just the well-known Groebner Basis of the given ideal $Id(PS)$ which has been found in some way different from the original one of Buchberger. Moreover, many known properties connected with the Groebner basis which are dispersed in the literature have been proved in some simple and unanimous manner. For details we refer to the author's paper [W2].

It turns out that the Russian mathematician Vladimir P.Gerdt has found also the Groebner basis of a polynomial ideal essentially in the same way as above. He has use an alternative name of *involutive basis* and has given also a detailed analysis of various possible notions of *multiplicativity* and *non-multiplicativity* due to Riquier, Janet, Thomas, and Gerdt himself. For more details we refer to the paper [G] of Gerdt.

References

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